GRAPH QUASI-DISTANCES AND EXTENSION OF SEMI-LIPSCHITZ FUNCTIONS IN MACHINE LEARNING (*WORK IN PROGRESS*)

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Abstract. Consider a topological graph G, where the topology is defined by a quasi-pseudo-metric. Suppose that a real valued function is defined in a quasi-pseudo-metric subspace of G representing a reward function for a given dynamical system in which the sequence of states is defined by the graph. Using well-known theoretical techniques for semi-Lipschitz functions, we extend the reward function to the whole graph. We use such procedure for providing a new iterative method of reinforced learning that can be applied to certain topological graphs, that are also presented.

1. Introduction and basic definitions

In this paper we propose a topological approach for reinforced learning on graphs endowed with a particular topology, where the topology is defined by means of a quasi-pseudo-metric. Our aim is to provide new models for understanding success reward systems is which there is an intrinsic asymmetry in the definition of the topology. This is the case, for example, when we want to introduce a directionality given by the time variable in evolutionary systems, or the case when the topology must reflect a hierarchical relation among the nodes of the graph. Indeed, the natural topological elements in a graph representing a dynamical system are non-symmetric: non-symmetric distances model in an optimal way the evolution of the system with the time. It must be said that a (symmetric) distance is often used for graphs: the so called path distance, defined as the infimum of the number of edges (maybe weighted) in all possible paths that connect different nodes of a graph (it can be found for example in [3]).

Thus, our arguments are based on the assumption of an underlying graph structure on the original space of states of the system, that allows to consider an adequate set of strategies on the space: a graph based analytical approach is considered. In this paper we establish a way for defining a family of quasi-pseudo-metrics for translating the graph to a topological space, becoming in this way what we call a quasi-pseudo-metric graph.

Nowadays there is an increasing interest in the study of minimal extensions of Lipschitz maps defined on graphs. This is an interesting topic, that was originally developed in the context of new methods related to reinforcement learning. The reader can find an updated survey on the topic in the first chapter of [9] (see also

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[4, 8]). However, the framework is restricted to Lipschitz extensions of functions acting in metric graphs. As we said, the problems that motivate our research are essentially non-symmetric, and so metrics must be substituted by quasi-metrics. In this direction, and in order to give support to our research, the theoretical setting for extending semi-Lipschitz maps in quasi-metric spaces have been recently investigated (see [6, 7, 10]). Applications of Lipschitz continuity for general metric spaces to reinforced learning can be found in [1] and the references therein. This opens the door to extend the research to this context.

Therefore, in this paper we are interested in finding extension theorems for semi-Lipschitz maps acting in topological subspaces of quasi-metric graphs, with the aim of developing a new method of reinforcement learning for artificial intelligence. In the first step we analyze some options for defining quasi-pseudo-metrics that would be adequate for modeling graphs constructed over sets of sequences of actions on a given system. In order to do that, we follow some theoretical tools related to the ones developed in [6, 7]. Concretely, we analyze the minimal extensions provided by the quasi-metric version of the McShane and the Whitney extension formulas for (semi)-Lipschitz maps. Interpolation among these situations could be interesting candidates for meaningful extensions of Lipschitz maps, that will be the starting point of our method.

We will present our results as follows. After the explanation of the basic graph structures, we will define the adequate quasi-pseudo-metrics on them in order to represent suitable topologies for the spaces. After presenting the main theoretical results, we will show some particular examples both for the discrete case and for the continuous one: the first one will deal with sequences of states and actions in a decision system (what we will call "a drunk man crossing a bridge"); the second one will represent a sequence of states in a financial market.

2. Preliminaries

A quasi-pseudo-metric is a function $d : M \times M \to \mathbb{R}^+$, where M is a set and such that for $a, b, c \in M$,

(1) $d(a, b) = 0$ if $a = b$, and
(2) $d(a, b) \leq d(a, c) + d(c, b)$.

($\mathbb{R}^+$ is the set of non-negative real numbers.) Such a function is enough for defining a topology by means of the basis of neighborhoods that is given by the open balls. If $\varepsilon > 0$, we define the ball of radius $\varepsilon > 0$ and center in $a \in M$ as

$$B_\varepsilon(a) := \Big\{ b \in \Omega : d(a, b) < \varepsilon \Big\}.$$

This topology is defined by the countable basis of neighborhoods provided by the balls $B_{1/n}(x) = \{y \in X : d(x, y) \leq 1/n\}$, $n \in \mathbb{N}$. The resulting metrical/topological structure (Ω, d) is called a quasi-pseudo-metric space.

If $d(a, b) = d(b, a)$ for $a, b \in M$, then it is called a pseudo-metric. If $d(a, b) = 0$ only in the case that $a = b$ it is called a quasi-metric, and if both requirements are satisfied, d is called a metric or a distance. Often the function d^{-1}, given by

$$d^{-1}(a, b) = d(b, a), \quad a, b \in M,$$

is defined: it is also a quasi-pseudo-metric, that is called the conjugate quasi-pseudo-metric. If d is a quasi-metric, we have also that

$$\max\{d(a, b), d^{-1}(a, b)\}, \quad a, b \in M,$$

is a metric.

The set of real Lipschitz functions acting in a metric space M is defined by such functions $f : M \to +$ that satisfy that

$$|f(a) - f(b)| \leq K d(a, b).$$

This definition fits with the case in which d is a metric, and in this case the Lipschitz constant of f is the infimum of all the constants K satisfying the inequality.

The problem of extending Lipschitz functions acting in subsets of graphs has been recently considered, both from the theoretical and the computational points of view (see [5, 9] and the references therein). In this research, however, the attention is centered in the case in which the topology defined in the graph is given by a (strict) metric, that is the one that is defined as what is called a weighted graph distance. It is defined for connected graphs as follows: a weight —a positive real number—is given to each edge of the graph; then the distance among the nodes a and b of the graph is defined as the infimum of all the sums of weights associated to each path that connect a and b. The aim of this paper is to introduce different topologies on the graph that are defined by non-symmetric functions, that is, by quasi-pseudo-metrics, that in general do not follow this kind of definition. Moreover, the research presented in [5, 9] do not use the McShane —or the Whitney— formula(s), since they are interested in finding extensions with higher order of smoothness.

One of the main tools that we use in our development is the extension of Lipschitz functions acting in subspaces of a metric space to the whole metric space. A classical result of the mathematical analysis establishes that this can always be done: indeed, the McShane-Whitney theorem says that if B is a (metric) subspace of a metric space (M, d) and $T : B \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant K, there is always a Lipschitz function $\tilde{T} : M \to \mathbb{R}$ extending T and with the same Lipschitz constant: that is, $\tilde{T}(a) = T(a)$ for all $a \in B$.

In particular, the function

$$\tilde{T}(x) := \sup_{a \in B} \{T(a) - K\, d(x, a)\}, \quad x \in M,$$

that is called the McShane extension formula, provides such an extension. We will use it for giving a constructive tool for our approximation. The Whitney formula, given by

$$T^W(x) := \inf_{a \in B} \{T(a) + K\, \rho(x, a)\}, \quad x \in M,$$

provides such an extension too.

We are interested in using such extensions for real functions acting in quasi-pseudo-metric spaces. The theoretical results extending the McShane-Whitney theorem to this situation are easy to prove in the same way that the original theorem, and were essentially presented in [6, 7]. As far as we know, these are the earliest references for these results. But the asymmetry of the quasi-metric functions change the results, that must be rewritten; a complete explanation can be found in [2]. We write the main result below for the aim of completeness. New definitions are needed.

Let (S, q) be a quasi-metric space. We say that a real function $f : S \to \mathbb{R}$ is semi-Lipschitz if there is a constant $K > 0$ such that for all $s, t \in S$,

$$\max\{\big(f(s) - f(t)\big), 0\} \leq K q(s, t).$$

The first article in which a systematic study of such operators was done —at least as far as we know— is [10].

Proposition 4.3 in [2] *(McShane extension for semi-Lipschitz maps)* Let (S,q) be a quasi-metric space, a subspace (S_0,q) and a semi-Lipschitz function $f: S_0 \to \mathbb{R}$ with constant $K>0$. Then the formula

$$\hat{f}(s) = \sup_{t \in S_0} \{f(t) - Kq(t,s)\}, \quad s \in S_0,$$

provides a semi-Lipschitz extension with the same constant K.

A c-semi-Lipschitz real function $f: S \to \mathbb{R}$ is a map that satisfies the inequalities

$$\max\{\big(f(s) - f(t)\big), 0\} \leq Kq(t,s), \quad s,t \in S.$$

We use this name because such a map is a semi-Lipschitz map in the conjugate space (S, d^{-1})

Proposition 4.4 in [2] *(Whitney extension for c-semi-Lipschitz maps)* Let (S,q) be a quasi-metric space, a subspace (S_0,q) and a c-semi-Lipschitz function $f: S_0 \to \mathbb{R}$ with constant $K>0$. Then

$$f^W(s) = \inf_{t \in S_0} \{f(t) + Kq(t,s)\}, \quad s \in S_0,$$

is a c-semi-Lipschitz extension with the same constant K.

3. Quasi-metric spaces of strategies and reinforcement learning

In this section we define a quasi-pseudo-metric structure for a dynamical system composed by discrete steps that define a sequence of either states or actions of the system. This will be the underlying structure in which we will implement a reward function that will be the main predictive tool of the model. Note that it has a natural graph structure provided by the time evolution.

3.1. Quasi-pseudo-metric spaces of strategies.

Let $\mathcal{U}$ be a set representing the states of a system and an associate space of strategies $\mathcal{S}$, that is the space of finite sequences of states endowed with a (quasi-pseudo)-metric. Each step in any sequence in $\mathcal{S}$ represents a change from a state in $\mathcal{U}$ to another state in $\mathcal{U}$. We call "actions" to a change from a state to another.

For defining such a metric we can consider several functions. The easiest way is to consider the discrete metric d in $\mathcal{U}$, that is, if $a,b \in \mathcal{U}$ we have that

$$d(a,b) = 1 \quad \text{if} \quad a \neq b \quad \text{and} \quad d(a,a) = 0 \quad \text{for all } a.$$

Then we can define a metric ρ in $\mathcal{S}$ as follows. If $s,v \in \mathcal{S}$,

$$\rho(s,v) = \sum_{i=1}^{\infty} w_{i,j} d(s_i, v_i) \quad \text{for all } s,$$

where s_i denotes the i-th coordinate of s, v_i denotes the i-th coordinate of v and $w_{i,j}$ are positive real numbers representing weights in the definition.

In the case that we consider the set of strategies based on the set of actions instead of the set of states of a system, the "discrete bifurcation metric" is more convenient. Let us explain this distance. The set $\mathcal{S}$ is defined by all the finite sequences of actions. It is based on the idea that a sequence of actions must represent a dynamics in a system. In fact, the elements of $\mathcal{S}$ can be understood as paths in a connected graph. A variation in an action of a sequence at a given point changes completely the states involved in the rest of the sequence. That is, two sequences s_1 and s_2 that coincides in the first three actions produce up to this point exactly the same states of a system. However, a change in the fourth position changes the state produced in the system, at this moment and until the end of the sequence, even if the rest of the coordinates in both sequences —from the fifth on—, are the same.

This motivates the definition of the following metric, that we denote by ρ_0. Consider the space of all finite sequences of actions with the following lattice operation: if $s = (s_1, s_2, ..., s_m)$ and $t = (t_1, t_2, ..., t_r)$, we define $s \wedge t$ as the sequence given by

$$s \wedge t = (s_1, s_2, ...s_n),$$

where n is the maximum value for which $s_1 = t_1,\ s_2 = t_2,\ ...\ s_n = t_n$.

Then we define

$$\rho_0(s, v) = \max\{length(s),\, length(v)\} \,-\, length(v \wedge s),$$

where $length(z)$ is the number of nontrivial coordinates of the sequence of actions z, and $z \wedge h$ is the sequence of actions defined by the first common coordinates of z and h.

Since we are interested in the construction of the most general analytic tool for including asymmetric effects, we will show in what follows that this metric can be defined as the maximum of a meaningful quasi-metric q and its conjugate q^{-1}.

We can define now a quasi-metric as follows. If $s, t \in \mathcal{S}$, we define.

$$q(s, t) = length(t) - length(s \wedge t);$$

in the same way, we can consider the conjugate quasi-metric given by

$$q^{-1}(s, t) = q(t, s) = length(s) - length(s \wedge t).$$

Note that, if t is defined as the first n coordinates of s, then $q(s, t) = 0$. Note also that

$$\max\{q(s, t), q^{-1}(s, t)\} = \max\{q(s, t), q(t, s)\} = \rho_0(s, t).$$

Let us prove below that q is indeed a quasi-metric, and so ρ_0 is a metric as a direct consequence.

Lemma 3.1. *Consider a set of strategies $\mathcal{S}$ defined by a set of actions. Then for every $r, s, t \in \mathcal{S}$, we have that*

$$q(r, t) \leq q(r, s) + q(r, s).$$

Consequently, and taking into account that the formula $\max\{q(s, t), q(t, s)\}$ defines a metric, we have that q is a quasi-metric.

Proof. By definition, we clearly have that

$$length(r \wedge t) \geq \min\{length(r \wedge s), length(s \wedge t)\}.$$

Thus we obtain that

$$length(r \wedge s) + length(t \wedge s) \leq length(r \wedge t) + length(s).$$

Therefore

$$q(r, t) = length(t) - length(r \wedge t)$$

$$\leq length(s) - length(r \wedge s) + length(t) - length(s \wedge t) = q(r, s) + q(s, t).$$

□

3.2. **Lipschitz maps in spaces of strategies.** Consider now a subset $\mathcal{B} \subseteq \mathcal{S}$. It represents the strategies that have been already checked, for which we already have an evaluation. That is, we can consider now an evaluation map, that is a real function $f : \mathcal{B} \to \mathbb{R}$ which, as we said, is supposed to be known. We consider it as a Lipschitz map. Since $\mathcal{B}$ is finite, the associated Lipschitz constant $K_B(f)$ is always finite.

It is well-known that we can always obtain a Lipschitz extension $\hat{f}$ of the evaluation function f to the whole space preserving the Lipschitz constant using a McShane type extension for Lipschitz operators. Thus, we extend the evaluation function to all the space of strategies. It can be already used to evaluate any strategy of the set, and so it provides a method for generating experience "to feed the system" from completely new situations that have not been checked in the real world. This is the main tool of our purpose of reinforced learning. Note that we can use either the metric ρ_0 or the quasi metrics q and q^{-1} for extending the reward function f, since we know the corresponding extension theorems for all these cases, as have been shown in Section 2.

The universe of states and the actions can be represented together as a directed graph in a standard way. Often this representation allow a clear picture of the problem (see for example [4, 6.1.1]). We will use such representation through the paper, since it also facilitates the use of graph-based analytic programs that can been chosen for our study.

4. A first simple example: the drunk man crossing the bridge

In this section we consider a Lipschitz extension of a Lipschitz function in a metric space. Consider a 3-state universe $\mathcal{U}$, and a set of 3 actions in them "going ahead" —go ahead right=1, go straight=2, go ahead left=3—. The set $\mathcal{S}$ of strategies is in this case defined by finite sequences of actions, that finish when some particular sequence of actions happen —the drunk man falls out of the bridge when he walks for example 3 subsequent times to the right—. We use the discrete bifurcation metric considered above. The evaluation function is given in this case by the length of the strategy. This of course can be computed for all the sequences, but we want to compare its values with the prediction of the evaluation function provided by the Lipschitz extension of this evaluation function based in its values for a little number of strategies.

After some experiments rolling a die, we get the following sequences of actions. (The symbol 0/0 in the following computations is evaluated as 0.)

$$H_1 = (2, 3, 2, 3), \quad H_2 = (1, 1), \quad H_3 = (1, 1),$$
$$H_4 = (2, 3, 3), \quad H_5 = (2, 3, 3), \quad H_6 = (1, 1),$$
$$H_7 = (1, 2, 1), \quad H_8 = (3, 3), \quad H_9 = (3, 1, 2, 3, 3, 3).$$

Thus, we have that

$$v(H_1) = 4, \, v(H_2) = 2, \, v(H_3) = 2 \, v(H_4) = 3, \, v(H_5) = 3,$$

$$v(H_6) = 2,\ v(H_7) = 3,\ v(H_8) = 2,\ v(H_9) = 6.$$

We will show how we can get an estimate $\hat{v}$ for v using the values of the first ones.

(1) Since one of them is repeated, we get a pseudo-distance instead of a distance with matrix
$$D = \begin{bmatrix} 0 & 4 & 4 \\ 4 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}.$$

(2) We compute the Lipschitz constant $K = \max_{i,k=1,2,3} \frac{|v(H_i)-v(H_k)|}{d(H_i,H_k)}$. We have that
$$\frac{|v(H_1)-v(H_2)|}{d(H_1,H_2)} = \frac{|4-2|}{4} = 1/2, \quad \frac{|v(H_1)-v(H_3)|}{d(H_1,H_3)} = \frac{2}{4} = 1/2, \quad \frac{|v(H_2)-v(H_3)|}{d(H_2,H_3)} = \frac{0}{0}.$$
Therefore, $K = 1/2$.

(3) Let us show the computation of the estimate for several H_i, $i > 3$.
- $\hat{v}(H_4) = \max_{k=1,2,3}\{v(H_k) - \frac{1}{2}d(H_k,H_4)\} = \max_{k=1,2,3}\{4-\frac{1}{2}2, 2-\frac{1}{2}3, 2-\frac{1}{2}3\} = 3$.
Comparing with the value of v, we get $v(H_4) = 3 = \hat{v}(H_4)$. The predictive value of the Lipschitz extension works in this case.
- $\hat{v}(H_6) = \max_{k=1,2,3}\{v(H_k) - \frac{1}{2}d(H_k,H_6)\} = \max_{k=1,2,3}\{4-\frac{1}{2}4, 2-\frac{1}{2}0, 2-\frac{1}{2}0\} = 2$.
Obviously in this case $\hat{v}(H_6) = v(H_6)$, since in fact $H_6 = H_2 = H_3$.
- $\hat{v}(H_7) = \max\{2,1,1\} = 2$. In this case, we get $\hat{v}(H_7) = 2$, while $v(H_7) = 3$. Thus, the values are not coinciding, but they are similar yet.
- $\hat{v}(H_9) = \max\{4-\frac{1}{2}6, 2-\frac{1}{2}6, 2-\frac{1}{2}6\} = 1$, while $v(H_9) = 6$. The values in this case are quite different.

5. An example involving quasi-metrics and semi-Lipschitz functions: the oracle function

We follow with the example provided in the previous section, but considering a new reward-type function and a genuine asymmetric structure of the topology. For improving the explanation of some concepts, We will use an extended notation by adding the character $\emptyset$ in each finite sequences in order to consider it an infinite sequence of characters. That is, we will write sometimes $(1,3,2,\emptyset,\emptyset,...)$ instead of $(1,3,2)$.

5.1. **The McShane semi-Lipschitz extension of the oracle function.** Let us define the oracle function acting in the set of strategies as follows. If $s = (s_1,...,s_n,\emptyset,...)$ is a complete strategy —that is, a strategy that have finished with the end of the process, so with a symbol $\emptyset$—, a sub-strategy is a sequence as $(s_1,...,s_r)$, for $r \leq n$. Consider a set $\mathcal{S}_0 \subseteq \mathcal{S}$ generated by a set $S_{0,0}$ of complete strategies by considering all the sub-strategies of all the elements of $S_{0,0}$. In the model, it is supposed that these strategies have been "experimentally" checked. The oracle function is a function $\mathcal{O}: \mathcal{S}_0 \to \mathbb{R}$ defined as

$$\mathcal{O}(s) = 0 \ \textit{if the next step-coordinate expected after } s \textit{ is } \emptyset$$

and $\mathcal{O}(s) = 1$ otherwise. In case of multiple values for a particular sub-strategy, we put the mean.

This function is intended to represent the possibility of surviving after the subsequence application of several actions: if after the application of a sequence of actions $s = (s_1, ..., s_n)$, the system is still "alive" —that is, $(s_1, ..., s_n)$ is not a complete strategy—, then the value is $\mathcal{O}(s) = 1$.

We are going to compute the semi-Lipschitz McShane extension of $\mathcal{O}$ for the particular subset of complete actions considered in Section 4. We consider the set

$$H_1 = (2,3,2,3), \quad H_2 = (1,1), \quad H_4 = (2,3,3)$$

that represent complete strategies. The set $\mathcal{S}_0$ is defined then as

$$\mathcal{S}_0 = \{(2), (2,3), (2,3,2), (2,3,2,3), (1), (1,1), (2,3,3)\}.$$

The values of $\mathcal{O}$ are:

$$\mathcal{O}((2)) = 1, \mathcal{O}((2,3)) = 1, \mathcal{O}((2,3,2)) = 1, \mathcal{O}((2,3,2,3)) = 0,$$
$$\mathcal{O}((1)) = 1, \mathcal{O}((1,1)) = 0, \mathcal{O}((2,3,3)) = 0.$$

The semi-Lipschitz constant is the maximum of the values as

$$\frac{\mathcal{O}((2)) - \mathcal{O}((2,3,2,3))}{q((2),(2,3,2,3))} = \frac{1}{3}, \ \frac{\max\{\mathcal{O}((2,3,2,3)) - \mathcal{O}((2)), 0\}}{q((2,3,2,3),(2))} = \frac{0}{0},$$
$$\frac{\mathcal{O}((2,3)) - \mathcal{O}((2,3,2,3))}{q((2,3),(2,3,2,3))} = \frac{1}{2}, \ \frac{\max\{\mathcal{O}((2,3,2,3)) - \mathcal{O}((2,3)), 0\}}{q((2,3,2,3),(2,3))} = \frac{0}{0},$$
$$\frac{\mathcal{O}((2,3,2)) - \mathcal{O}((2,3,2,3))}{q((2,3,2),(2,3,2,3))} = \frac{1}{1} = 1, \ \frac{\max\{\ \mathcal{O}((2,3,2,3)) - \mathcal{O}((2,3,2)), 0\}}{q((2,3,2,3),(2,3,2))} = \frac{0}{0}.$$

It is easy to see that the semi-Lipschitz constant equals 1, and this is the case for every set $\mathcal{S}_0$ constructed as above. Let us show the values of $\hat{\mathcal{O}}$ —the semi-Lipschitz extension of $\mathcal{O}$— when acting in any other sequence.

Consider $H = (2,1,1)$. We have that

$$\hat{\mathcal{O}}(H) = \max_{s \in \mathcal{S}_0}\{\mathcal{O}(s) - q(s,H)\} = \max\{1-2, 1-2, 1-3, 1-2, 0-3, 0-2, 0-3, 0\} = 0.$$

In general, it can be noted that by the definition of $\hat{\mathcal{O}}$, the only elements H such that $\hat{\mathcal{O}}(H) = 1$ are the ones that are (strictly) subsequences of a sequence in the original set $\mathcal{S}_{0,0}$, that is, the elements of $\mathcal{S}_0$ that are not complete actions.

5.2. **The Whitney semi-Lipschitz extension of the oracle function.** The formula that gives the extension in this case is,

$$\mathcal{O}^W(s) = \inf_{t \in S_0}\{f(t) + Kq(t,s)\}, \quad s \in S_0,$$

However, we cannot apply it; using the setting that we shown in the example of the previous section, we have that

$$\frac{\mathcal{O}((2)) - \mathcal{O}((2,3,2,3))}{q((2,3,2,3),(2))} = \frac{1-0}{1-1} = \frac{1}{0},$$

what means that the oracle function $\mathcal{O}$ cannot be considered as a c-semi-Lipschitz function. Following [10, p.294], we can say that the reason is that this function is not $\leq_{q^{-1}}$-increasing.

6. Applications: investments in a financial market

Let us show an application to the automatic short-range business management of investments in a financial market. Consider a set of 6 companies listed on a financial market. In a sequence of times —for instance, each day in two weeks—, an investor is allowed to sell the ones he got in the last step and to buy new ones. In each step, he can only buy shares of one of the six companies that are considered.

Then, we consider sequences of 10 daily selling/buying events corresponding to two weeks of financial activity. That is, the elements of our metric space are vectors of Ω^{10}. Let us write the complete model —similar to the previously explained "drunk man crossing a bridge" problem—, including an evaluation function that is defined by the increase/decrease of the monetary value of the shares of each type at each step.

The set of states that is considered is given by a six-element set $\Omega = \{a, b, c, d, e, f\}$; one of them can be defined to mean "no investment state". We consider the symmetric version of the "discrete bifurcation metric" ρ_0 for this example. Recall that it is given by

$$\rho_0(s, v) = \max\{length(s), length(v)\} - length(v \wedge s), \quad s, v \in \Omega^{10}.$$

At each step $i \in \{1, ..., 10\}$ the evaluation function v_i for a state $w \in \Omega$ is the value of a share of w if it is sold at the time of the step i. The incremental evaluation function Δv_i is defined as

$$\Delta v_i(w) = \frac{v_i(w) - v_{i-1}(w)}{v_{i-1}(w)}, \quad w \in \Omega, \;\; i \in \{1, ..., 10\}.$$

At each i order now the elements of Ω by the order of the real numbers $\Delta v_i(w)$, $w \in \Omega$. We define now the order function O_i at each step i for a state w as the order number of w with respect to the ordering given above, starting by 0. For example, if $\Delta v_3(a) = 0.3$, $\Delta v_3(b) = -0.1$, $\Delta v_3(c) = 0$, $\Delta v_4(d) = -0.2$, $\Delta v_3(e) = 0.35$, $\Delta v_3(f) = -0.01$, the values of O_i are

$$O_3(a) = 4, \;\; O_3(b) = 1, \;\; O_3(c) = 3 \;\; O_3(d) = 0, \;\; O_3(e) = 5, \;\; O_3(f) = 2.$$

The success function E is defined in this example as follows. If s is a sequence belonging to Ω^{10}, we define

$$E(s) = \sum_{i=1}^{10} O_i(s_i).$$

This gives an estimate of the relative success of the strategy represented by the sequence of states s for a given experience.

Other estimate would be given for example by

$$R(s) = \sum_{i=1}^{10} \Delta v_i(s_i).$$

that is more connected with the numeric value of the success of the investment given by s; note however that this does not give the increment of the amount of money after the investment. We will consider the firstly proposed E.

With all these elements, the algorithm for getting the McShane extension would be the following.

(1) A complete two weeks experience with the corresponding functions v_i is supposed to be known. Compute all the values of the functions Δv_i.
(2) Using it, write the table of all the values of O_i.
(3) For a subset $\mathcal{S}_0$ of strategies, compute the values of the estimate E.
(4) Compute the Lipschitz constant K for E and the set $\mathcal{S}_0$, that is given by

$$K = \max\{\frac{|E(s) - E(t)|}{\rho_0(s,t)} : s, t \in \mathcal{S}_0\}.$$

Here, d is the path metric associated to the quasi-metric q and given by $\rho_0(s,t) = \max\{q(s,t), q(t,s)\}$ explained in the previous sections.
(5) Compute the McShane extension by

$$\hat{E}(s) = \sup_{r \in S_0} \{E(r) - Kd(r,s)\}.$$

(6) Compare the values of the extension $\hat{E}$ with the true values of E for elements $s \in \mathcal{S}$ that are not in $\mathcal{S} \setminus \mathcal{S}_0$.

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