# QUASI-PSEUDO-METRICS AND EXTENSION OF LIPSCHITZ-TYPE FUNCTIONS IN MACHINE LEARNING *(WORK IN PROGRESS)*

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Abstract. Consider a set $\mathcal{U}$ of states of a dynamical system. We define a quasi-metric space $M$ whose elements are sequences of states, that in certain cases can be considered as a quasi-metric space. A reward Lipschitz function that is defined in a subspace $M_0$ of $M$ is supposed to be known. Using both McShane and Whitney type new extensions of semi-Lipschitz operators acting in quasi-pseudo-metric spaces, we find a reward function that is defined for all the elements of $M$. This provides a new theoretical framework for reinforced learning.

## 1. Introduction and basic definitions

The extraordinary amount of data provided by the new information technologies suggests that some aspects of the fundamentals of machine learning should be changed. For example, data describing the behavior of "success reward systems" —as financial markets—, are updated in very short periods of time, what provides a huge source of information. Methods for dealing with such kind of data should be based on solid mathematical structures and should be as simple as possible, in order to get fast and reliable results.

Our fundamental ideas are based on the known technology for extending Lipschitz functions acting in subspaces of quasi-pseudo-metric space. Extension of Lipschitz functions defined in such subspaces of metric spaces have already shown their usefulness, at least from the theoretical point of view (see for example [1, 11]; see also [8, 12] for the specific case of Lipschitz functions defined in metric graphs). However, as far as we known the asymmetry in the definition of the topology has not been explored yet. although it seems to be highly relevant for the modeling of directed structures, as directed graphs, Markov chains or general dynamical systems.

The aim of this paper is to explain a new topological framework for reinforced learning in the context of quasi-pseudo-metric spaces. In the last years a meaningful amount of related papers have been published, apparently in separate settings. The nature of the problems that constitute our original motivation is essentially non-symmetric, that is metrics must be substituted by quasi-metrics. As far as we know, this new framework has not been taken into account in the new developments. However, there are theoretical research papers that allow to easily go on with the necessary generalizations to the non-symmetric setting. For example, in a series of

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papers that were published at the beginning of the century, theoretical questions regarding semi-Lipschitz maps —the natural version of Lipschitz maps when they act in quasi-metric spaces—, were studied (see for example [13]). The papers [9, 10] present already some central result regarding semi-Lipschitz extensions of semi-Lipschitz maps acting in quasi-metric spaces. More recent papers have followed this research line, still from the theoretical point of view.

Our ideas are presented in the paper as follows. After a general explanation of the topologies defined by quasi-pseudo-metrics, we center our attention in showing the constructive method that proves the classical McShane-Whitney extension of Lipschitz maps to quasi-pseudo-metric spaces. After that, we show several examples of how our technique can be applied, mainly for the case of analyzing the state of financial markets.

## 2. Preliminaries

Let us introduce some notation and concepts. We will write $\mathbb{R}^+$ for the set of non-negative real numbers. A quasi-pseudo-metric on a set $M$ is a function $d : M \times M \to \mathbb{R}^+$ such that for $a, b, c \in M$,

(1) $d(a, b) = 0$ if $a = b$, and
(2) $d(a, b) \leq d(a, c) + d(c, b)$.

Such a function is enough for defining a topology by means of the basis of neighborhoods that is given by the open balls. If $\varepsilon > 0$, we define the ball of radius $\varepsilon$ and center in $a \in M$ as

$$B_\varepsilon(a) := \Big\{ b \in M : d(a, b) < \varepsilon \Big\}.$$

Note that this topology is in fact given by the countable basis of neighborhoods provided by the balls $B_{1/n}(x) = \{y \in X : d(x, y) \leq 1/n\}$, $n \in \mathbb{N}$. The resulting metrical/topological structure $(M, d)$ is called a quasi-pseudo-metric space.

If the function $d$ is symmetric, that is, $d(a, b) = d(b, a)$, then it is called a pseudo-metric. If $d$ separates points —that is, if $d(a, b) = 0$ only in the case that $a = b$— but it is not necessarily symmetric, then it is called a quasi-metric. Finally, if both requirements hold —symmetry and separation—, $d$ is called a metric (or a distance). In this case, the topology generated by the balls is Hausdorff.

We will deal with topologies associated to quasi-pseudo-distances that are defined using a real function (typically, an reward function). Next example show how to construct such quasi-pseudo-metrics.

**Example 2.1.** *The exponential and logarithmic quasi-metrics. Let $(M, q)$ be a quasi-pseudo-metric space an consider a function $v : M \to \mathbb{R}$. Throughout the paper such a function will be called an evaluation function or a reward function —depending on the context—. Define the function $\Gamma : M \times M \to \mathbb{R}^+$ as follows. If $a, b \in M$,*

$$\Gamma(a, b) = q(a, b) + \big| e^{-v(a)} - e^{-v(b)} \big|.$$

*It is easy to see that this is a quasi-pseudo-metric. The function*

$$\Delta(a, b) = q(a, b) + \log(1 + \big| v(a) - v(b) \big|)$$

*is also a quasi-pseudo-metric. Both of them can be used for considering a topology for the space in which the neighborhoods are weighted by the evaluation of a function; in the first one, the topology gives that two points are near if both of them have* *good an similar evaluations, and far if they have bad but different evaluation. The behavior of the second one is the opposite.*

## 3. Extension of Lipschitz functions to maximal metric domains

Let $(M, \rho)$ be a metric space. A function $f : M \to \mathcal{R}$ is a Lipschitz function if there is a positive constant $K$ such that

$$|f(s) - f(v)| \leq K d(s, v).$$

The infimum of such constants $K$ is called the Lipschitz constant $K_M(f)$.

We will deal with extension of Lipschitz functions defined on a subset of a metric space to the whole metric space. The McShane-Whitney theorem states the following: if $B$ is a subset of a metric space $(M, \rho)$ and $T : B \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant $K$, there always exists a Lipschitz function $\tilde{T} : M \to \mathbb{R}$ extending $T$ and with the same Lipschitz constant. The function

$$\tilde{T}(x) := \sup_{u \in B} \{T(u) - K\,\rho(x, u)\}, \quad x \in M,$$

provides such an extension; it is sometimes called the McShane extension. We will use it for giving a constructive tool for our approximation. Alternatively, the Whitney formula, given by

$$T^W(x) := \inf_{u \in B} \{T(u) + K\,\rho(x, u)\}, \quad x \in M,$$

provides also an extension.

Other relevant extension theorem in the setting of the Hilbert spaces is the celebrated Kirszbraun theorem. It establishes that given $H$ and $K$ Hilbert spaces, $B$ is a subset of $H$ and $T : B \to K$ a Lipschitz operator, then there is another Lipschitz operator $\tilde{T} : H \to K$ that extends $T$ and has the same Lipschitz constant as $T$ (see [7], [14, p.21]). This result will not be used here.

## 4. Metric spaces of strategies and Lipschitz maps: the reinforced learning method

Consider a set $\mathcal{U}$. An action on it is a map $\varphi : \mathcal{U} \to \mathcal{U}$.

1) Let $\mathcal{U}$ be a set representing either the possible states of a system or a set of actions that can be done on these states. For the aim of simplicity we will consider now that this is a set of possible states. We assume that it contains the empty set as an element of $\mathcal{U}$ which represents that no state has happened yet. Let us define the associated space of strategies $\mathcal{S}$. It is the space of finite sequences of states, endowed with a (quasi-pseudo)-metric. Each step in any sequence in $\mathcal{S}$ represents a change of state.

let us define now a quasi-pseudo-metric on $\mathcal{S}$. In order to do this we can consider different functions. In the first step, if we assume that we have not previous information, it is defined as the discrete metric $d$ in the set of states $\mathcal{U}$. If $a, b \in \mathcal{U}$ we have that

$$d(a, b) = 1 \quad \text{if} \quad a \neq b \quad \text{and} \quad d(a, a) = 0 \quad \text{for all } a.$$

An extension of the same idea, consisting in giving positive weights depending on $a$ and $b$ different to 1, can also be used.

Using this, we define the metric $\rho$ in $\mathcal{U}$ as follows. If $s, v \in \mathcal{U}$,

$$\rho(s,v) = \sum_{i=1}^{\infty} d(s_i, v_i) \quad \text{for all } s,$$

where $s_i$ denotes the $i$-th coordinate of $s$.

Other (quasi-pseudo) metrics can be used, as will be shown later on. A relevant one consists on a weighted sum of the cosinus angular pseudo-metric and a norm, if we are dealing with vectors.

2) Consider a (finite) subset $\mathcal{B} \subseteq \mathcal{S}$. It represents the strategies that have been already checked, for which we already have an evaluation (reward function). That is, we can consider now an evaluation map, that is a real function $f : \mathcal{B} \to \mathbb{R}$ which, as we said, is supposed to be known. We consider it as a Lipschitz map. Since $B$ is finite, the associated Lipschitz constant $K_B(f)$ is always finite.

It is well-known that we can always obtain a Lipschitz extension $\hat{f}$ of the evaluation function $f$ to the whole space preserving the Lipschitz constant using a McShane type extension for Lipschitz operators. Thus, we extend the evaluation function to all the space of strategies. It can be already used to evaluate any strategy of the set.

**Example 4.1.** *If we assume that we have (typically) a "value" function $V$ for each state of the system when it appears in a certain coordinate of position $i$ of any strategy, we can use it to define a quasi metric as explained in Example 2.1. That is, for a state $a \in \mathcal{U}$ appearing in the position $i$ of any strategy, we have a value (a real number) that we denote by $V(a,i)$. If $s_1$ and $s_n$ are the first and the last states of a strategy $s$, we define $f(s) = V(s_n, n) - V(s_1, 1)$ and we use the formulas in Example 2.1.*

3) In the next step, we can use this extension to define a "success reward" quasi-metric $\Gamma$ for the space of strategies. Adding new information, that is, checking more strategies and adding the corresponding evaluations to the existing ones, we can develop a recursive method of reinforced learning. We will show the algorithm in the next section.

As we said in Example 2), the "success reward" quasi-metric can be given for example by the function $\Gamma$ defined in Example 2.1. This means that, for a fixed $\epsilon > 0$, a successful strategy $s$ has a reduced number of strategies in the $\epsilon$-ball $B_\epsilon(s)$ —the ones that are close to improve it—, while $B_\varepsilon(v)$ contains a big number —also the ones that improve it—. So, the topological meaning is clear: "good" strategies have small neighborhoodhoods, and "bad" ones has big neighborhoodhoods.

**Remark 4.2.** *The universe of states and the actions can be represented together as a directed graph in a standard way. Often this representation allows a clear picture of the problem (see for example* [3, 6.1.1]*). We will use such representation through the paper, since it also facilitates the use of graph-based analytic programs that can been chosen for our study.*

4.1. **Extension of semi-Lipschitz functions in quasi-metric spaces.** The arguments for extending Lipschit functions defined in subspaces preserving the Lipschitz constant that are known for metrics can be extended for quasi-pseudo-metric spaces easily. The main tool for doing this is to prove that the McShane-Whitney Theorem works also in this case, and the McShane formula can be used for extending the corresponding Lipschitz-type map. This fact is essentially already known; the reader can find related results in [9, 10] (see also [13]). For the aim of completeness, we include here the easy proof, that follows the same scheme that the one for the McShane-Whitney Theorem.

Let $(S,q)$ be a quasi-metric space. We say that a real function $f : S \to \mathbb{R}$ is semi-Lipschitz if there is a constant $K > 0$ such that for all $s,t \in S$,

$$\max\{\big(f(s) - f(t)\big), 0\} \leq Kq(s,t).$$

(See for example [13]).

In the same way, we can introduce the conjugate definition. Recall that, if $q$ is a quasi-metric on $S$, the conjugate quasi-metric $q^{-1}$ is defined as

$$q^{-1}(s,t) = q(t,s), \quad s,t \in S.$$

According to this, we say that a real function $f : S \to \mathbb{R}$ is c-semi-Lipschitz if there is a constant $K > 0$ such that for all $s,t \in S$,

$$\max\{\big(f(s) - f(t)\big), 0\} \leq Kq^{-1}(s,t) = Kq(t,s).$$

That is, *$f$ is c-semi-Lipschitz if and only if it is semi-Lipschitz with respect to the conjugate quasi-norm $q^{-1}$.*

**Proposition 4.3.** *(McShane extension for semi-Lipschitz maps) Let $(S,q)$ be a quasi-metric space, a subspace $(S_0,q)$ and a semi-Lipschitz function $f : S_0 \to \mathbb{R}$ with constant $K > 0$. Then the formula*

$$\hat{f}(s) = \sup_{t \in S_0} \{f(t) - Kq(t,s)\}, \quad s \in S_0,$$

*provides a semi-Lipschitz extension with the same constant $K$.*

*Proof.* 1) Note first that, if $s,t \in S_0$, we have that

$$f(s) - K\,q(s,s) \leq \sup_{t \in S_0} \{f(t) - Kq(t,s)\} = \hat{f}(s).$$

On the other hand, we have that $f(t) - f(s) \leq Kq(t,s)$, for all $t \in S_0$ and so

$$f(t) - Kq(t,s) \leq f(s).$$

This gives $\hat{f}(s) \leq f(s)$, and so $\hat{f}(s) = f(s)$.

2) Let us show now that $\hat{f}$, that is a well-defined function, is semi-Lipschitz. Take $s,t \in S$. If $\hat{f}(s) \leq \hat{f}(t)$ there is nothing to prove. Otherwise,

$$\begin{aligned} \hat{f}(s) - \hat{f}(t) &= \sup_{z \in S}\{f(z) - Kq(z,s)\} - \sup_{w \in S}\{f(w) - Kq(w,t)\}| \\ &\leq \sup_{z \in S}(f(z) - Kq(z,s) - f(z) + Kq(z,t)) \\ &\leq K\,q(s,t). \end{aligned}$$

This gives the result.

□

Alternatively, the non symmetric version of Whitney formula can also be given for getting an extension of a c-semi-Lipschitz map acting in a quasi-metric space. Thus, we get also the next

**Proposition 4.4.** *(Whitney extension for semi-Lipschitz maps) Let $(S,q)$ be a quasi-metric space, a subspace $(S_0,q)$ and a c-semi-Lipschitz function $f:S_0\to\mathbb{R}$ with constant $K>0$. Then*

$$f^W(s)=\inf_{t\in S_0}\{f(t)+Kq(t,s)\},\quad s\in S_0,$$

*is a c-semi-Lipschitz extension with the same constant $K$.*

*Proof.* 1) If $s,t\in S_0$, we have that

$$f(s)=f(s)+K\,q(s,s)\geq\inf_{t\in S_0}\{f(t)+Kq(t,s)\}=f^W(s).$$

We also have that $f(s)-f(t)\leq Kq^{-1}(s,t)$, for all $t\in S_0$ and so

$$f(t)+Kq^{-1}(s,t)=f(t)+Kq(t,s)\geq f(s)$$

for all $t\in S_0$. This gives $f^W(s)\geq f(s)$, and so $f^W(s)=f(s)$.

2) We show now that $f^W$ is a c-semi-Lipschitz function. Take $s,t\in S$. If $f^W(s)\leq f^W(t)$ there is nothing to prove. Otherwise, writing $z_0$ for the element of $S$ for which the infimum in the definition of $f^W(t)$ is attained, we obtain

$$f^W(s)-f^W(t)\leq f(z_0)+Kq(z_0,s)-f(z_0)-Kq(z_0,t)\leq Kq(t,s)=Kq^{-1}(s,t).$$

The proof is done.

□

## 5. Application: a reward function for the states of a financial market

Let su show in this section a direct application of the Lipschitz extension of reward functions, in the context of the mathematical models in economy. We fix a financial market of 4 different companies $A,B,C,D$ and $E$, and a sequence of differente times in a bigger time period.

1) *The associated metric space.* The values of the events in the market are taken from the real behavior of the market: at each time point, we compute the next state of the system by writing the improvement or loose of the value of the shares of each company. Than is, suppose that the values of the of the shares at the time $t=3$ are

$$v_3(A)=3,\quad v_3(B)=2,\quad v_3(C)=4,\quad v_3(D)=1,\quad v_3(E)=0$$

and in the next moment $t=4$ we have that the values are

$$v_4(A)=1,\quad v_4(B)=1,\quad v_4(C)=4,\quad v_4(D)=2,\quad v_4(E)=0.$$

Then the state 4 is given by the vector $s=(-2,-1,0,1)$, corresponding to the differences of values of each company: $\Delta A=-2$, $\Delta B=-1$, $\Delta C=0$, $\Delta D=1$ and $\Delta E=0$.

Consider then the set $M_0=\{s_i: i=1,...,50\}$, where a set of states —5-coordinates vectors— associated to a real sequence of times ($t=0,t=1,...,t=40$, for example). Note that some of these states may coincide:

then they are considered to be the same, since we take a subset of four coordinates vectors. The distance among these elements that we consider is related with the cosinus similarity; however, the associated cosinus distance cannot be directly used since it does not satisfy the distance axioms. Using the cosinus,

$$Cos(s_i, s_j) = \frac{s_i \cdot s_j}{\|s_i\| \, \|s_j\|},$$

we define a distance by mixing the angle

$$\Theta(s_i, s_j) = \frac{ArcCos\big(\frac{s_i \cdot s_j}{\|s_i\| \, \|s_j\|}\big)}{\pi},$$

and an Euclidean component

$$E(s_i, s_j) = \|s_j - s_i\|_2 = \sqrt{\sum_{k=1}^{4} |s_i^k - s_j^k|^2}.$$

We define then the distance for our space as

$$d_\epsilon(s_i, s_j) = \Theta(s_i, s_j) + \epsilon E(s_j, s_i). \tag{5.1}$$

We can use for defining the distance $d$ for example $\epsilon = 1/10$, that is, we use

$$d(s_i, s_j) = d_{1/10}(s_i, s_j) = \Theta(s_i, s_j) + \frac{1}{10} E(s_j, s_i).$$

We consider $(M_0, d)$ as a subspace of a bigger metric space $(M, d)$, where $M$ is the space of all the possible sequence of 5-coordinates.

2) *The reward function that we want to extend.* We are interested in giving an extension to $M$ of the real valued function $R$ acting in $M_0$ that represents the reward that can be associated to each state. A state $s_i \in M_0$ gives an idea of the behavior of the market in a given situation. The reward function computed for a given state $s$ is given by

$$R(s) = \sum_{i=1}^{4} s^i, \quad s = (s^1, s^2, s^3, s^4).$$

Note that it has the meaning of an index of "how succesful" is a given state.

3) *MacShane and Whitney extension formulas.* First we have to compute the Lipschitz constant $K$ for the reward function $R$ in order to get the extension $\hat{R}$, for which the same $K$ works. The extension is used for computing two global indices that give an idea of how favourable the market is for investing.

**Remark 5.1.** *Although the reward function can be computed for any other state, we use the experience given by the value of such function for the "real states" in order to know the conditions of the real behavior of the market. Thus, $\hat{R}$ represents the state of the market, and projects this state on each possible new situation that would be given.*

Let us show now *a concrete application: how to use $\hat{R}$ to evaluate the "state of the market".*

We define now an index to evaluate the state of the market using the prediction of $R$ given by $\hat{R}$. In order to do this, we consider a density type parameter that can be defined using this new notion. We assume that the states of $M_0$ have been checked in the real market, and all of them are possible in a market that have not changed its properties (that is, the stability and trend properties of the market have not evolved: essentially they do not depend on the time. Take a radius $r > 0$ (for instance r=1/10) and consider the topological space given by all the elements of $M_0$. If we have trained the system enough, we can consider that any new situation would be provided by a new state $s$ that should be "near" the reference states; we can model it by defining a neighbor that contais the unio of balls of all the elements in $M_0$ of a given radius, as

$$\mathcal{B}_0 \subseteq \bigcup_{s \in M_0} B_r(s).$$

Although the evaluation function of the states that are considered in the set $\mathcal{B}_0$, we assume that this evaluation is not adequate. Actually, in coherence with our construction the success of a state of the market —that is, $R$— that is close to $M_0$ must be measured in terms of the similarity of the new state with the elements of $M_0$, and not using the procedure that we have used for computing $R$ in $M_0$. Thus, we use $\hat{R}$ for defining the parameter.

We choose an evaluation function $\phi$ that is defined as a density function as follows. Consider Lebesgue measure $\mu$ in $\mathbb{R}^4$. Then

$$\phi(r, M_0) = \frac{\int_{\mathcal{B}_0} \hat{E}(s)\, d\mu(s)}{\mu(\mathcal{B}_0)}.$$

This gives an idea of how favorable the choice of investing in the market is, taking into account that success in investing is ultimately a probabilistic event. For the effective computing, this integral can be approximated by using the estimate of $\hat{E}$ in a finite set of points.

Additionally, the behavior of the limit

$$\lim_{r \to \infty} \phi(r, M_0),$$

would also give some information about the system not depending on the size of the neighborhood that are considered, for $\mathcal{B}_0$ defined exactly as $\bigcup_{s \in M_0} B_r(s)$ and so varying also with $r$.

Let us show some computations. Take a set of initial states for which we have a reward function that can be computed for a given subset of verified real data of a market with five products; the empty product is considered also, to give the chance to the system to not to invest in a given step. For them we have a reward function that is defined for each of the states, that is not necessarily known for states that has not been observed in the real sequence of states. However, to simplify we compute it as the sum of the increases/decreases of the value of all the five products at each step, but in general it can only be computed for situations —states— that have really happened.

For the application of the previous model for a concrete case, let us show a choice of 9 consecutive states that have been obtained from a experience in a market of 5 products. These states are

$$s_1 = (1, 1, 2, 3, 1), \quad s_2 = (1, 2, 4, 1, 2), \quad s_3 = (4, 1, 3, 1, 2),$$
$$s_4 = (1, -2, 0, 2, 1), \quad s_5 = (1, 1, -2, 0, 1), \quad s_6 = (1, 4, 0, 0, -2),$$
$$s_7 = (1, 0, 0, -2, 1), \quad s_8 = (1, -2, 0, 1, -1), \quad s_9 = (1, -4, 0, -1, 3).$$

These states define the set $M_0$. The metric in the space is $d_{1/5}$, which is defined in Equation (5.1). We use a reward function that can be directly computed for these vectors by the formula

$$R(s_i) := \sum_{j=1}^{5} s_i^j, \quad s_i = (s_i^1, ..., s_i^5), \quad i = 1, ..., 5.$$

Recall that such a function cannot be in general computed by such a simple equation. It must contain the evaluation of the decision maker, based on his experience, or any other information he could consider for improving the reward of a given state. Therefore, the reward cannot be computed for other possible states using a similar formula, since the context surrounding other possible states have not occurred in the "real" world.

In the following step, we consider a set of 30 randomly generated states in a 5-dimensional hipercube. We assume that this hipercube is an adequate neighborhood of the set of states that we have considered. Two extensions of the reward function $R$ are considered, the McShane extension —giving a minimal value—, and the Whitney extension —which provides a maximal function—. Figure 5 provides a representation of these reward functions for the initial set $M_0$ and the other randomly generated 30 states. The reader can notice that the McShane extension —shown on the left side of the figure— gives a downward estimate, while the Whitney extension —show in the right side— provides an upward estimate.

Both of them may be used for giving general evaluation indexes of the "state of the market". The idea, as was explained above, is to give an estimate of the density of the reward function in the chosen neighborhood of the set $M_0$, that is in this case a 5-dimensional hipercube. The formula given before can be directly used.

We use the 30 random states to give a Monte-Carlo estimate of both densities. The results are

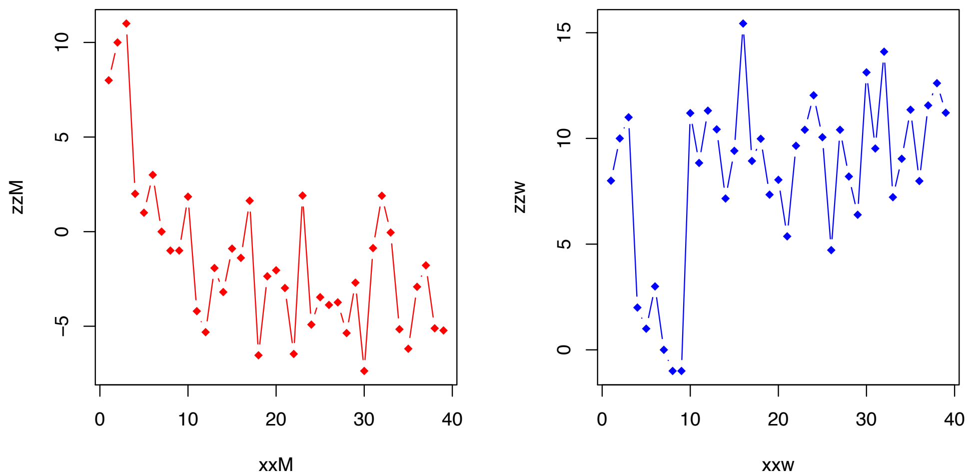
*Density for the McShane extension=* $-1.428569$.

*Density for the Whitney extension=* $8.360589$.

These values are relatively stable, as was verified by repeating the computations with sets of different size of randomly generated vectors, as well as different —but similar— neighborhoods. For example, the following sequence of estimates of the corresponding densities was obtained by sorting a number $t$ (multiple of 30) of random states: the variation is small when $t$ is increasing.

$$t = 60, \; McShane = -2.318583, \; Whitney = 8.900048$$
$$t = 90, \; McShane = -2.163690, \; Whitney = 9.124692$$
$$t = 120, \; McShane = -2.566929, \; Whitney = 9.220755$$
$$t = 150, \; McShane = -2.658777, \; Whitney = 9.231013$$

Figure 1. Representation of the McShane and the Whitney extensions of the reward function $R$.




$$t = 180,\ McShane = -2.944394,\ Whitney = 9.377209$$
$$t = 210,\ McShane = -3.049436,\ Whitney = 9.621343$$
$$t = 240,\ McShane = -2.709648,\ Whitney = 9.543604$$
$$\dots$$
$$t = 3000,\ McShane = -3.268723,\ Whitney = 9.81852$$
$$\dots$$
$$t = 30000,\ McShane = -3.286915,\ Whitney = 9.750687$$
$$\dots$$
$$t = 100000,\ McShane = -3.259854,\ Whitney = 9.747255$$

Comparing both values and taking into account the definition of the reward function for the set $M_0$, the decision maker can use these values as a test of the state of the market: it seems to be favorable, since the positive index is clearly bigger than the negative one. Note also that this conclusion does not depend on the size of $t$.

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